J. Fluid Mech. (1976), *vol.* 74, *part* 1, *pp.* 161–182
Printed in Great Britain

Large amplitude lunate-tail theory of fish locomotion

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(Received 7 July 1975)

The two-dimensional theory of lunate-tail propulsion is extended to motions of arbitrary amplitude, regular or irregular, so that an accurate comparison may be made with the actual lunate-tail propulsion of scombroid fishes and cetacean mammals. There is no restriction at all on the amplitude of motion but the tail's angle of attack relative to its instantaneous path through the water is assumed to remain small. The theory is applied to the regular finite amplitude motion of a thin aerofoil with a rounded leading edge to take advantage of the suction force and a sharp trailing edge to ensure smooth tangential flow past the rear tip. This can represent the vertical motions of the horizontal lunate tails of large aspect ratio with which cetacean mammals propel themselves or the horizontal undulations of the vertical lunate tails of certain fast fishes. The dependence of the thrust, the hydromechanical propulsive efficiency and the energy wasted in churning up the eddying wake on the reduced frequency, the angle of attack and the amplitude of motion is exhibited.

1. Introduction

The study of the problem of the locomotion of animals in general, and marine organisms in particular, includes as a major component an analysis of how the animals can elicit a propulsive force by oscillatory motions in the surrounding medium. Part of the work done in those motions is used for balancing the body's viscous resistance whereas the rest is wasted in churning up the fluid in the eddying wake. It is obvious that for efficient swimming the rate of working should be as close as possible to the thrust, which balances the resistance of the water, times the cruising speed. Lighthill's (1969, 1970, 1971, 1975) fundamental studies of animal locomotion, carried out with great hydrodynamical brilliance and zoological insight, bring out the important concept of the hydromechanical propulsive efficiency, which is akin to the Froude efficiency of a propeller and defined as the ratio of the mean rate of working of the forward thrust to the mean rate of work done by the movements of the body and its appendages on the surrounding fluid. The advantages of increasing this parameter may have constituted one of the important guiding factors in the evolutionary development of swimming animals. This parameter depends on the propulsive mode, which, following Breder (1926), can be broadly classified as anguilliform or carangiform.

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These categories blend into one another, and should not be thought of as sharply distinct. In addition, there are some equally interesting but, perhaps, not so efficient modes of propulsion such as pectoral-fin swimming, jet propulsion etc.

Undulatory propulsion, in which a transverse wave of increasing amplitude passes along the body from head to tail, is found in animals swimming with relatively low hydrodynamical efficiency. Ciliary and flagellar propulsion of micro-organisms, swimming at very low values of the Reynolds number R based on the animal's length and the forward velocity, the motion of almost all the animals swimming at low to moderate R ($\leqslant 10^3$) and the motion of bad hydromechanical shapes, whose cross-sectional form is not such as to enhance the virtual-mass effect, swimming at high R fall in this category. The theory dealing with the locomotion of these animals was named 'resistive' by Lighthill (1971) as the force between a small section of the animal and the surrounding water is a resistive force which depends mainly on the instantaneous value of the velocity of that section relative to the water. Hancock (1953) generalized Taylor's (1952) small amplitude analysis for low Reynolds numbers to undulatory motions of arbitrarily large amplitude by an ingenious method of distribution of Stokeslets and source doublets along the instantaneous centre-line of the organism at each instant.

In carangiform propulsion, the amplitude of undulation can be small or even zero in the anterior half or two-thirds of the body, increasing posteriorly to a large value near the trailing edge with a phase lag. This mode of swimming, a development of the anguilliform mode, relies more on the reactive forces which can always operate at a better efficiency and is found in a wide variety of fishes and other vertebrates. Animals using the carangiform mode produce sudden accelerations in the water and the resultant thrust is derived from the reactive rate of change of fluid momentum. Lighthill (1960, 1970), using the ideas of slender-body theory, developed the elongated-body reactive-force theory, which is applicable to the majority of the fishes using the carangiform mode of propulsion, and was able to establish that higher speeds and efficiency can be achieved with certain morphological alterations, viz., necking of the body anterior to the caudal fin and maintenance of large depth of the cross-section near the mass centre, needed for reducing recoil due to unbalanced oscillations of the side-force. The large amplitude elongated-body theory was developed by Lighthill (1971) by an appropriate distribution of source doublets along the spinal column of the fish, and was effectively used by Weihs (1972) in analysing Gray's (1968) data on fish turning.

All the fast marine animals, namely certain scombroid fishes, certain sharks and all the cetacean mammals, have adopted essentially the carangiform mode of propulsion, their tails having converged to a crescent-moon shape of high aspect ratio through different pathways of evolution in the pursuit of high hydromechanical propulsive efficiency. Profound necking anterior to the lunate tail imparts to it the character of a propeller to which all the undulations are confined. This mechanism of swimming may be termed 'lunate-tail swimming propulsion' as it is characteristically found in the animals having lunate tails.

Lighthill's (1960, 1971) elongated-body theory cannot be applied here as its fundamental assumption, that the disturbances are produced by body actions distributed along the body axis, breaks down. The action of the lunate tail is spread out at right angles to the direction of motion: vertically in the case of fishes and horizontally in the case of cetacean mammals. A keener look at the lunate tail shows that it is like an aeroplane wing, with every chordwise section resembling an aerofoil having a rounded leading edge and a sharp trailing edge, which suggests that the techniques of unsteady wing theory can be applied to study lunate-tail hydrodynamics. Either of the following two methods can be used to analyse the problem.

(i) Start with the hydrodynamical equations and make use of potential theory.

(ii) Construct the solution using a distribution of vortices or sources.

Lighthill (1970) made a start on the study of lunate-tail hydrodynamics by the first method and based his analysis on Prandtl's concept of the acceleration potential, which had earlier been brilliantly applied in Possio's (1940) work on the motion of aerofoils in compressible media and Wu's (1961) work on the swimming efficiency of thin plates. Chopra (1974) based his three-dimensional analysis of unsteady lunate-tail hydrodynamics on the vorticity distribution and found it convenient for representing the effect of the streamwise wake vorticity, resulting from the finiteness of the span, which was neglected in Lighthill's (1970) two-dimensional theory. The analysis gave a clear picture of the dependence of the forward thrust and the hydromechanical efficiency on the complete range of values of the aspect ratio, the feathering parameter and the reduced frequency besides the important confirmation that the pitching axis has to be approximately along the trailing edge for best efficiency. In spite of its generality the study is inadequate for providing an accurate comparison with lunate-tail propulsion because Fierstine & Walters' (1968) experiments on the wavyback skipjacks show that the amplitude (the distance between the maximum lateral excursion of a given point along the tail and the axis of progression measured perpendicular to the axis of progression) of the tail movement is of the order of two chord lengths, large enough to invalidate the results of small amplitude theory.

The object of this paper is to work out a two-dimensional reactive-force theory which could be applied to large amplitude lunate-tail propulsion. This analysis is complementary to Hancock's (1953) 'large amplitude resistive theory' and Lighthill's (1971) 'large amplitude elongated-body theory'. Heaving and pitching motion is considered. This characterizes the tail flukes of cetacean mammals such as whales and dolphins. The same motions turned through a right angle will represent the oscillations of the lunate tails of fishes. The analysis is carried out for motion of arbitrary amplitude, regular or irregular, and the general expressions for the thrust, the power required to maintain motion, the energy imparted to the wake and the hydromechanical propulsive efficiency are determined for this general motion as functions of the physical parameters defining the problem: namely the reduced frequency, the path amplitude and the angle of incidence.

Unsteady motion of an aerofoil is accompanied by shedding of the boundary layer from the trailing edge in the form of a thin vortex sheet, and its intensity is determined here from the integral equation obtained from the law of conservation of circulation. Subsequently, the complete vorticity distribution of the system is worked out using Wagner's (1925) concepts and the theory of impulses is applied to find the forces and moment experienced by the aerofoil. The hydromechanical swimming efficiency is analysed from work and energy considerations.

Finally, a case of regular finite amplitude motion of the aerofoil is discussed in detail and the theory is subjected to several checks by way of comparisons with Lighthill's (1970) small amplitude theory. Some representative curves are given to show the dependence of the thrust coefficient (mean forward thrust per unit area scaled on static pressure) and the hydromechanical efficiency on the path amplitude, the angle of attack and the reduced frequency.

An assumption that simplifies all the analysis is that, although the heaving motions are of large amplitude, the accompanying pitching motions are such that the effective angle of attack (the angle between the plane of the lunate tail and its direction of motion through the water) remains relatively small. As a result, the vortices shed are relatively weak and remain fairly close to where they are shed. The modest amount of available observational data, coupled with the likelihood of serious flow separation if this condition were not satisfied, seems to make this assumption a reasonable one.

2. Vorticity distribution

The theory is set out in a frame of reference in which the water far from the swimming animal is at rest. The Y axis is vertically upwards, the $-X$ axis along the constant mean direction of swimming and the Z axis is directed to make $OXYZ$ a right-handed co-ordinate system.

To describe the large amplitude motions of the lunate tail, a moving right-handed co-ordinate system $oxyz$ is attached to the tail with the origin at the trailing edge, the $x-$axis along the mean chord and y axis perpendicular to the tail in the upward direction (figure 1). Let the path of the trailing edge be $X = X_0(t)$, $Y = Y_0(t)$. Then the direction of motion of the trailing edge is $\theta(t)$, where $\tan\theta(t) = \dot{Y}_0(t)/\dot{X}_0(t)$ and a dot denotes differentiation with respect to time. Let the angle of attack (the angle between the tail's direction of motion and the tail's surface, assumed small) be $\alpha(t)$. The inclination of the tail to the X axis is then given by $\theta(t)-\alpha(t)$. Transformation from the co-ordinate system fixed in the tail to the co-ordinate system fixed in the fluid at rest can be effected through

$$\left.\begin{aligned} X-X_0(t) &= x\cos(\theta-\alpha)-y\sin(\theta-\alpha), \\ Y-Y_0(t) &= y\cos(\theta-\alpha)+x\sin(\theta-\alpha). \end{aligned}\right\} \tag{1}$$

The tail itself is $y = 0$, $-2c < x < 0$, and for fixed x on the tail the absolute velocity $(\dot{X}, \dot{Y})$ is given by

$$\dot{X} = \dot{X}_0(t)-x(\dot{\theta}-\dot{\alpha})\sin(\theta-\alpha),$$
$$\dot{Y} = \dot{Y}_0(t)+x(\dot{\theta}-\dot{\alpha})\cos(\theta-\alpha).$$

This absolute velocity thus has a component

$$u_1 = \dot{X}\cos(\theta-\alpha)+\dot{Y}\sin(\theta-\alpha)$$

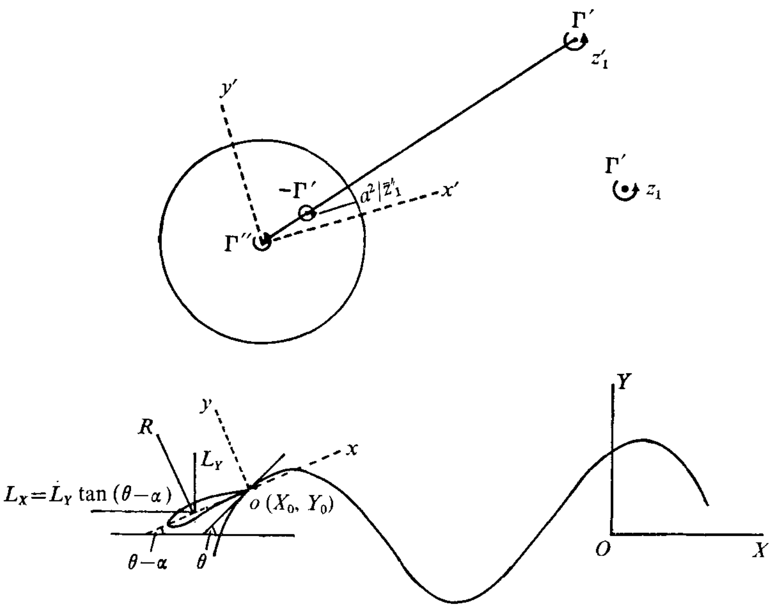

FIGURE 1. Co-ordinate systems and conformal representation of tail profile and a wake vortex for finite amplitude motion.

tangential to the tail, in the x direction, and a component

$$v_1 = \dot{Y}\cos(\theta-\alpha) - \dot{X}\sin(\theta-\alpha)$$

perpendicular to the tail. These components, on substituting for $\dot{X}$ and $\dot{Y}$ and making use of the fact that $\alpha(t)$ is small, simplify to

$$u_1 = \dot{X}_0 \sec\theta, \quad v_1 = A + xB, \tag{2), (3}$$

where

$$A = \alpha \dot{X}_0 \sec\theta, \quad B = \dot{\theta} - \dot{\alpha}.$$

By assuming that the flow about the lunate tail is two-dimensional, that the movement of any part of the tail from the path described by the trailing edge is small and that the vortices shed from the trailing edge lie along the path of the trailing edge, the analysis can be carried out without excessive complications. The complex-conjugate velocity $dW/dz = \partial\phi/\partial x - i\,\partial\phi/\partial y$ for the absolute motion of the fluid exterior to the profile of the body is given by Cauchy's formula:

$$\frac{dW}{dz}\left(\frac{z+2c}{z}\right)^{\frac{1}{2}} = \frac{1}{2\pi i}\oint \frac{\dfrac{dW}{d\zeta}\left(\dfrac{\zeta+2c}{\zeta}\right)^{\frac{1}{2}}}{\zeta - z}\, d\zeta, \tag{4}$$

which, making use of (3) and the fact that

$$\left(\frac{z+2c}{z}\right)^{\frac{1}{2}} = \begin{cases} i\left(\dfrac{x+2c}{-x}\right)^{\frac{1}{2}} & \text{for} \quad y \to +0, \\[2ex] -i\left(\dfrac{x+2c}{-x}\right)^{\frac{1}{2}} & \text{for} \quad y \to -0, \end{cases}$$

becomes

$$-i\left(\frac{2c+x}{-c}\right)^{\frac{1}{2}}\{u(x,-0)-iv(x,-0)\} = \frac{i}{\pi}\int_{-2c}^{0}\frac{A+xB}{\xi-x}\left(\frac{2c+\xi}{-\xi}\right)^{\frac{1}{2}}d\xi.$$

The clockwise vortex intensity $\gamma_0(x)$ on the aerofoil is given by

$$\gamma_0(x) = -2u(x,-0) = 2\left(\frac{-x}{x+2c}\right)^{\frac{1}{2}}\{A+B(x+c)\}. \tag{5}$$

The total quasi-steady vorticity Γ_0 (the vorticity which would be produced, according to Munk's (1922, 1924) thin-aerofoil theory, by the motion of the aerofoil or by the given velocity distribution in the air if the wake had no effect) can be obtained by integrating the vortex intensity $\gamma_0(x)$ from the leading edge to the trailing edge, viz.

$$\Gamma_0 = \pi c(2A-cB). \tag{6}$$

However, as the total circulation around the aerofoil is variable, because of the non-uniformity of the motion, an equivalent amount of oppositely directed vorticity must be shed from the trailing edge in the form of a wake composed of continuously distributed vortex lines, and the interference from this wake must be taken into account. The vortex intensity of the wake is determined from the following two conditions.

(i) The total vorticity of the whole system is invariant, or equivalently the rate of change of the circulation around the aerofoil is equal to the rate of vortex shedding.

(ii) The vortices can be regarded as remaining at the places where they are shed.

The effect of an element of vorticity Γ' located at a point in the wake may be evaluated by the method of conformal transformation as shown in figure 1. The aerofoil can be mapped onto a circle in the z' plane through

$$z+c = z'+a^2/z' \quad \text{with} \quad a = \tfrac{1}{2}c, \tag{7}$$

in which case the region outside the aerofoil in the z plane is transformed into the region outside the circle in the z' plane. Here and later on z and z' are used to denote $x+iy$ and $x'+iy'$ as there is no likelihood of confusion because the flow is considered to be two-dimensional.

Any point vortex Γ'' situated at the origin of the z' plane has streamlines which are concentric circles with centre at the point where the vortex is located. It is well known that a point vortex can be introduced at any other point of the circular region without disturbing the streamlines if a second vortex with circulation of the same magnitude but opposite sign is placed at the inverse point. Thus the vortices Γ' at z_1' and $-\Gamma'$ at $a^2/\bar{z}_1'$ along with any point vortex Γ'' at the centre of the circle will make the circle a streamline.

The complex potential $W(z')$ due to this vortex distribution (figure 1) is given by

$$W(z') = \frac{i\Gamma'}{2\pi}\left[\log\left(z'-\frac{a^2}{\bar{z}_1'}\right)-\log(z'-z_1')\right]-\frac{i\Gamma''}{2\pi}\log z'.$$

Also

$$\oint \frac{dW}{dz}\,dz = \oint (u\,dx + v\,dy) + i\oint (u\,dy - v\,dx).$$

The real part of this is the circulation and the imaginary part multiplied by ρ, the density of the fluid, is the mass flux across the contour, which is zero in our case as there are no sources present, and therefore

$$\Gamma = \oint \frac{dW}{dz'}\,dz' = \Gamma'' - \Gamma'.$$

Γ'' can be obtained from the Kutta condition of tangential flow at the trailing edge as follows:

$$u - iv = \frac{dW}{dz} = -\frac{i}{2\pi(1-a^2/z'^2)}\left[\Gamma'\left\{\frac{1}{z'-z_1'} - \frac{1}{z'-a^2/\bar{z}_1'}\right\} + \frac{\Gamma''}{z'}\right],$$

where $\bar{z}'$ is the complex conjugate of z'. On the aerofoil, which is a circle in the z' plane, by setting $z' = a e^{i\theta}$

$$u - iv = -\frac{1}{4\pi a \sin\theta}\left[\Gamma'' - \Gamma'\frac{z_1'\bar{z}_1' - a^2}{(a e^{i\theta} - z_1')(a e^{-i\theta} - \bar{z}_1')}\right]. \tag{8}$$

At the trailing edge $\theta = 0$, there will be a singularity unless

$$\Gamma'' = \Gamma'\frac{z_1'\bar{z}_1' - a^2}{(a - z_1')(a - \bar{z}_1')},$$

where z_1' is known in terms of z_1 through

$$z_1' = \tfrac{1}{2}\{(z+c) + [(z+c)^2 - c^2]^{\frac{1}{2}}\}. \tag{9}$$

The total vorticity induced by a point vortex Γ' located at z_1 is given by

$$\Gamma = \Gamma'(\alpha_1/\beta - 1), \tag{10}$$

where $\alpha_1 = x_1'^2 + y_1'^2 - a^2$ and $\beta = x_1'^2 + y_1'^2 + a^2 - 2ax_1'$. The vortex intensity induced on the aerofoil by the wake vortex Γ' is given by

$$\gamma_1(x) = u_+ - u_-,$$

where u_+ and u_- stand for the velocity on the upper and lower sides of the aerofoil. Substituting for u_- from (8) and simplifying we get

$$\gamma_1(x) = \frac{\Gamma'}{2\pi[c^2 - (c+x)^2]^{\frac{1}{2}}}\frac{\alpha_1}{\beta}\left\{2 - \beta\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\right\}, \tag{11}$$

where

$$\delta = x_1'^2 + y_1'^2 + a^2,$$
$$\tau_1 = \delta - (x+c)x_1' - y_1'[c^2 - (c+x)^2]^{\frac{1}{2}},$$
$$\tau_2 = \delta - (x+c)x_1' + y_1'[c^2 - (c+x)^2]^{\frac{1}{2}}.$$

The total induced vorticity Γ given by (10) can also be obtained by integrating (11) from $-2c$ to 0. For a rectilinear wake (10) and (11) reduce to

$$\Gamma = \Gamma'\left\{\left(\frac{x_1 + 2c}{x_1}\right)^{\frac{1}{2}} - 1\right\}$$

and $$\gamma_1(x) = \frac{1}{\pi}\frac{\Gamma'}{(x_1-x)}\left(\frac{-x}{x+2c}\right)^{\frac{1}{2}}\left(\frac{x_1+2c}{x_1}\right)^{\frac{1}{2}},$$

which agree with equations (8) and (7) of von Kármán & Sears (1938).

The vorticity Γ_1 induced by the complete wake can be calculated with the help of (10) if we set $\Gamma' = \gamma(x_1)\,[(dy_1/dx_1)^2+1]^{\frac{1}{2}}$ and integrate over the wake, viz.

$$\Gamma_1 = \int_0^l \gamma(x_1)\left\{\frac{\alpha_1}{\beta}-1\right\}\left[\left(\frac{dy_1}{dx_1}\right)^2+1\right]^{\frac{1}{2}} dx_1,$$

where dy_1/dx_1 is the slope of the path of the trailing edge referred to oxy and l is the abscissa of the furthest point of the wake. The circulation about the aerofoil at any instant is given by $\Gamma_0+\Gamma_1$. The vorticity Γ_w in the wake is given by

$$\Gamma_w = \int_0^l \gamma(x_1)\left[\left(\frac{dy_1}{dx_1}\right)^2+1\right]^{\frac{1}{2}} dx_1,$$

and the conservation of circulation gives

$$\Gamma_0+\int_0^l \gamma(x_1)\frac{\alpha_1}{\beta}\left[\left(\frac{dy_1}{dx_1}\right)^2+1\right]^{\frac{1}{2}} dx_1 = 0.$$

Referring to the fixed co-ordinate system, this can be transformed into

$$\Gamma_0+\int_{X_0}^{l_X} \gamma(X_1)f(X_1,X_0)\sec\theta_1\,dX_1 = 0, \tag{12}$$

where $\theta_1 = \tan^{-1}\{dY_1/dX_1\}$ and $f(X_1,X_0) = \alpha_1/\beta$ can be calculated for each value of X_1 through (1) and (9). This integral equation of the first kind, which has a kernel with a square-root singularity, can be solved for the wake vortex intensity, which in turn through (11) gives the vortex intensity induced on the aerofoil. The calculation of the forces and moment experienced by the lunate tail and the energy wasted in the wake, based on the impulsive momentum of the vortex system, is carried out in the next section.

3. General expressions for thrust, wake energy and efficiency

The sum of the strengths of all the vortices, on the tail and in the wake, is zero, as in the case of a vortex pair, and therefore the theory of the 'impulse' can be applied. Let us consider a domain D of the X, Y plane bounded by a contour C_1 which encloses the aerofoil as well as the wake. For this region

$$\iint_D \gamma(X,Y)\,dX\,dY = 0,$$

or $$\int_{C_1}(u\,dX+v\,dY) = 0,$$

which ensures the existence of a single-valued velocity potential $\phi(X, Y)$ outside C_1, and the component of the impulse in the Y direction, Q say, is given by

$$Q = \oint_{C_1}\rho\phi(X,Y)\,n_Y\,dS, \tag{13}$$

where dS is an element of C_1 and n_Y is the Y component of a unit inward normal to C_1. Using the facts that $n_Y = dX/dS$ and that ϕ is single valued, (13) becomes

$$Q = \oint_{C_1} \rho[X(u\,dX + v\,dY)].$$

By shrinking the contour C_1 to coincide with the aerofoil and the wake and accounting for the discontinuity in the velocity components tangential to the aerofoil and the vortex wake, the following expression for Q can be obtained:

$$Q = \rho \int_{X_0 - 2c\cos(\theta-\alpha)}^{X_0} X\gamma(X)\sec(\theta-\alpha)\,dX + \rho\int_{X_0}^{l_X} X_1\gamma(X_1)\sec\theta_1\,dX_1,$$

where l_X is the X co-ordinate of the furthest point of the wake and X and X_1 are used to express the abscissae of points on the aerofoil and in the wake respectively. By following a similar analysis, the expressions for the X component P of the impulse and the moment M_m of the impulse about the mid-chord position of the aerofoil can be obtained as

$$P = -\rho \int_{X_0 - 2c\cos(\theta-\alpha)}^{X_0} Y\gamma(X)\sec(\theta-\alpha)\,dX - \rho\int_{X_0}^{l_X} Y_1\gamma(X_1)\sec\theta_1\,dX_1$$

and

$$M_m = \frac{\rho}{2}\int_{X_0 - 2c\cos(\theta-\alpha)}^{X_0} [X^2 + Y^2 - 2XX_c - 2YY_c]\,\gamma(X)\sec(\theta-\alpha)\,dX$$
$$+ \frac{\rho}{2}\int_{X_0}^{l_X} [X_1^2 + Y_1^2 - 2X_1X_c - 2Y_1Y_c]\,\gamma(X_1)\sec\theta_1\,dX_1,$$

where $$X_c = X_0 - c\cos(\theta-\alpha), \quad Y_c = Y_0 - c\sin(\theta-\alpha).$$

Euler's formulae give

$$L_X = -dP/dt, \quad L_Y = -dQ/dt \tag{14), (15}$$

and

$$M = -dM_m/dt, \tag{16}$$

where (L_X, L_Y) denotes the force per unit span experienced by the lunate tail and M is the moment per unit span about its mid-chord position.

We want expressions for the forward thrust that will be uniformly valid for large and for small heaving amplitudes. In the case of small amplitudes, the forward thrust, being a second-order effect, cannot be accurately determined from (14) as in our calculation of P we have neglected second-order effects by assuming that the vortices remain at the positions where they are shed. But we can make use of the vertical force, which is a first-order effect. Furthermore, its value when multiplied by $\tan(\theta-\alpha)$ does give one important component of the thrust (see figure 1). However, another important contribution to the forward thrust results from the suction force associated with the fast flow around the leading edge. This suction force takes instantaneously the well-known steady-flow value and through Blasius's formula works out to be

$$-\pi\rho|K|^2,$$

where $K(x+2c)^{-\frac{1}{2}}$ is the asymptotic form of $\frac{1}{2}[\gamma_0(x) + \gamma_1(x)]$ as $x \to -2c$ ($\gamma_0(x)$ and $\gamma_1(x)$ are the vorticities induced by the angle of attack and the wake respectively). The value of K can be obtained using (5) and (11), and the suction force experienced by the lunate tail per unit span works out to be

$$F_s = -2\pi\rho c\left\{A - Bc + \frac{1}{\pi}\int_0^l \gamma(x_1)\frac{\alpha_1}{\beta}\frac{ax_1'}{(\delta+cx_1')}\left[\left(\frac{dy_1}{dx_1}\right)^2+1\right]^{\frac{1}{2}}\frac{dx_1}{c}\right\}^2. \tag{17}$$

Referred to the fixed co-ordinate system

$$F_s = -2\pi\rho c\left[A - Bc + \frac{1}{\pi}\int_{X_0}^{l_X} \gamma(X_1) f_1(X_1, X_0)\sec\theta_1 dX_1\right], \tag{18}$$

where
$$f_1(X_1, X_0) = \frac{\alpha_1}{\beta}\frac{ax_1'}{(\delta+cx_1')} = \frac{ax_1'\alpha_1}{\delta^2 - 4a^2x_1'^2}$$

is known for each value of X_1 through (1) and (9), and the total forward thrust (positive when directed in the $-X$ direction) per unit span of the lunate tail is given by

$$L_X = F_s\cos(\theta-\alpha) + L_Y\tan(\theta-\alpha).$$

The mean forward thrust $\bar{L}_X$ per unit span thus becomes

$$\bar{L}_X = \frac{1}{(l_X - X_0)}\int_{X_0}^{l_X} L_X dX. \tag{19}$$

The power required to maintain the motion is equal to the rate E at which work is done by the lunate tail, which from the principles of work can be written as

$$E = L_X\dot{X}_0 + L_Y\dot{Y}_0 + M(\dot{\theta}-\dot{\alpha}),$$

whereas the mean rate of working per unit span is given by

$$\bar{E} = \frac{1}{(l_X - X_0)}\int_{X_0}^{l_X} E\, dX. \tag{20}$$

The hydromechanical propulsive efficiency, defined as the ratio of the mean rate at which useful work is done to the overall mean rate of working, can be obtained from

$$\text{efficiency} = \dot{X}_0\bar{L}_X/\bar{E}. \tag{21}$$

The swimming efficiency can also be obtained, however, by the method preferred below using overall energy-balance considerations. These, according to inviscid flow theory, assert that the power input or the rate of doing work E is equal to the rate $L_X\dot{X}_0$ at which work is done by the forward thrust and that the kinetic energy lost to the fluid, which is equivalent to the wake energy, is given by

$$T = -\tfrac{1}{2}\rho\int_{X_0}^{l_X}\psi\gamma(X_1)\sec\theta_1 dX_1,$$

where ψ is the solution of the Poisson equation

$$\nabla^2\psi = \gamma(X_1),$$

viz.
$$\psi = \frac{1}{2\pi}\int_{X_0}^{X_l}\gamma(X_2)(\log r)\sec\theta_2 dX_2,$$

where $\gamma(X_2)$ denotes the value of γ at (X_2, Y_2) and r stands for

$$\{(X_1 - X_2)^2 + (Y_1 - Y_2)^2\}^{\frac{1}{2}}.$$

Thus
$$T = -\frac{1}{4\pi}\rho \int_{X_0}^{l_X}\int_{X_0}^{l_X} \gamma(X_1)\,\gamma(X_2)\log(r)\sec\theta_1\sec\theta_2\,dX_2\,dX_1.$$

The extra energy shed into the wake per unit time by the undulations of the aerofoil is given by

$$W_E\,dt = -\frac{1}{4\pi}\rho \int_{X_0}^{X_0+\dot{X}_0\,dt} \gamma(X_1)\left\{\int_{-\infty}^{\infty} \gamma(X_2)\log r \sec\theta_2\,dX_2\right\}\sec\theta_1\,dX.$$

So the mean rate $\overline{W}_E$ at which kinetic energy is imparted to the fluid becomes

$$\overline{W}_E = \frac{1}{(l_X - X_0)}\int_{X_0}^{l_X} W_E\,dX$$

and the hydromechanical efficiency of swimming can be obtained from

$$\text{efficiency} = \dot{X}_0\bar{L}_X/(\dot{X}_0\bar{L}_X + \overline{W}_E). \tag{22}$$

As the trailing vortex sheet is continuously extending, neither W_E nor $\overline{W}_E$ can be negative.

This lunate-tail theory, developed for motion of arbitrary amplitude, is now applied to regular motion of cetacean mammals with horizontal tail flukes. It also applies to a variety of fast fishes having relatively large and rigid bodies with vertical caudal fins of the order of $\frac{1}{12}$ to $\frac{1}{10}$ of the total length (the distance in a straight line between the most anterior projection of the snout and the tip of the longest lobe of the caudal fin), viz. the wahoo, skipjack, tunny, louvar, etc. from the Percomorphi and whale sharks, porbeagles etc. from the Selachii, which undulate their crescent-moon-shaped fins symmetrically about the caudal peduncle without exhibiting any appreciable bending.

4. Thrust and efficiency resulting from regular finite amplitude swimming

The finite amplitude theory developed in the preceding section is applied to the specific problem of the estimation of the thrust and efficiency obtained by the above-mentioned animals when swimming at constant speed through oscillations of their lunate tails such that the trailing edge describes a sinusoidal path $Y_0 = h_1 \cos \nu t$, where h_1 is the amplitude of the path and ν its radian frequency. The variation of the angle of attack depends on many factors, such as the muscular power applied to the fin, the velocity of the fin and consequently the resistance of the water and the structural strength of the fin. For simplicity we assume that a 90° phase difference exists between the sinusoidally varying angle of attack $\alpha = \alpha_0 \sin \nu t$ and the path. Within the limitations of the reactive-force theory of thrust production, we study the balance between the mean thrust, produced reactively by the transverse motions of the lunate tail, and the mean drag, produced resistively by the longitudinal motions. As the fish is swimming with constant speed U with its axis of progression along the $-X$ axis,

$$X_0 = -Ut$$

and therefore the direction θ of the trailing edge is given by

$$\tan\theta = -h\omega\sin(\omega X_0/c),$$

where $h = h_1/c$ (amplitude of the path scaled on the semi-chord) and $\omega = \nu c/U$ is the reduced frequency (the ratio of the time taken to travel a distance of π chords to the time taken to complete one oscillation). The reduced frequency was measured to be 0·45 by Fierstine & Walters (1968) during their careful experiments on the skipjack, which shows that the unsteady effects cannot be neglected. The integral equation (12) in this particular case becomes

$$D_1 = \int_{X_0}^{l_X} \frac{\gamma(X_1)}{U} f(X_1, X_0) \sec\theta_1 \frac{dX_1}{c}, \tag{23}$$

where $\qquad D_1 = -\pi\{\omega\alpha_2\cos\omega X_0/c - 2\alpha\sec\theta\},$

with $\qquad \alpha_2 = (\alpha_0 - \omega h\cos^2\theta).$

It can be seen that the kernel $f(X_1, X_0)$ of the integral equation has a square-root singularity at the trailing edge X_0 and cannot be differentiated with respect to the lower limit to yield an integral equation of the second kind, as was demonstrated by Mangler (1951). If the motion is supposed to have continued for a long time $l_X \to \infty$, and also, because of the periodic oscillations of the lunate tail, the vorticity shed into the wake is periodic with zero mean. Accordingly we can write (23) as

$$D_1 = \int_{X_0}^{X_0+X_P} \frac{\gamma(X_1)}{U} \sum_{n=1}^{\infty} [f(X_1 + (n-1)XP, X_0)] \sec\theta_1 \frac{dX_1}{c} + \Gamma w, \tag{24}$$

where $X_P = 2\pi c/\omega$ and Γ_w accounts for the fluctuations in the wake vorticity. If the wake consists of a finite integral number of periods $\Gamma_w = 0$ while if the wake extends beyond a finite integral number of periods Γ_w is non-zero, periodic in X_0 with zero mean and given by

$$d\Gamma_w/dX_0 = -\gamma(X_0)\sec\theta. \tag{25}$$

$\gamma(X_1)\sec\theta_1/U$, being periodic, can be expanded as a Fourier series with zero mean, viz.

$$\frac{\gamma(X_1)\sec\theta_1}{U} = \sum_{m=1}^{\infty} [a_m\cos(m\omega X_1) + b_m\sin(m\omega X_1)],$$

which can in turn be used in (25) to yield

$$\Gamma_w = -\sum_{m=1}^{\infty}\left\{\frac{a_m}{m\omega}\sin(m\omega X_0) - \frac{b_m}{m\omega}\cos(m\omega X_0)\right\}.$$

In (24) we have been able to convert the infinite range of integration to a finite one, over one period only, using the periodicity of the wake vorticity but the following two difficulties still remain to be overcome before we can apply the routine process for solving integral equations of the first kind.

(i) The kernel $f(X_1, X_0)$ is not converging.

(ii) The kernel has a square-root singularity at the lower limit of integration.

The first problem is overcome by subtracting a suitable constant value (varying appropriately, however, with the number of periods taken into account) from the kernel as this does not affect the final result but ensures fast convergence of the kernel. The constant is the limit of the kernel as $X_1 \to \infty$ and can be obtained as follows:

$$f(X_1, X_0) = \frac{x_1'^2 + y_1'^2 - a^2}{x_1'^2 + y_1'^2 + a^2 - 2ax_1'} = 1 + \frac{2a}{x_1'} + O\left(\frac{1}{x_1'^2}\right).$$

From (9) it can be seen that $x_1' \to x_1$ as $x_1' \to \infty$ and from (1) $x_1 \to X_1 \cos(\theta - \alpha)$. Therefore

$$f(X_1, X_0) = 1 + \frac{2a}{X_1 \cos(\theta - \alpha)} + O\left(\frac{1}{X_1^2}\right).$$

The constant value to be subtracted from the kernel derived from this analysis is

$$1 + \frac{2a}{nX_P \cos(\theta - \alpha)} = 1 + \frac{\omega \sec(\theta - \alpha)}{2n\pi},$$

which decreases as X_1 increases.

The square-root singularity at the lower limit of integration is removable and can be taken care of by the standard procedure of transformation of the variable of integration according to

$$X_1 = X_0 + u^2.$$

These modifications convert (24) into

$$\tfrac{1}{2}D_1 = \int_0^{(X_P)^{\frac{1}{2}}} u g_1(X_0 + u^2) \left\{ \sum_{n=1}^{\infty} \left[g_2(X_0 + u^2) - \frac{\omega \sec(\theta - \alpha)}{2n\pi} \right] \right\} du + \frac{\Gamma_w}{2}, \quad (26)$$

where $\quad g_1(X_0 + u_2) = \sum_m [a_m \cos\{m\omega(X_0 + u^2)\} + b_m \sin\{m\omega(X_0 + u^2)\}]$

and $\quad g_2(X_0 + u^2) = f(X_0 + u^2 + (n-1)X_P, X_0) - 1.$

The left-hand side of (26) is known in terms of X_0 and each value of X_0 gives an algebraic equation in the Fourier coefficients, whose number can be so chosen that the convergence of

$$\frac{\gamma(X_1)}{U} \sec\theta_1 = \sum_m \{a_m \cos(m\omega X_1) + b_m \sin(m\omega X_1)\} \quad (27)$$

is ensured; for example $m = 10$ gives values of $\gamma(X_1)/U$ correct to four decimal places for the values of the amplitude taken up in our analysis. A typical coefficient a_m is

$$-\frac{\sin(m\omega X_0)}{2m\omega} + \int_0^{(X_P)^{\frac{1}{2}}} u \cos\{m\omega(X_0 + u^2)\} \sum_{n=1}^{\infty} \left[g_2(X_0 + u^2) - \frac{\omega \sec(\theta - \alpha)}{2n\pi} \right] du,$$

wherein the value of the integral is obtained by Simpson's rule after the convergent numerical values of the series and subsequently of the integrand have been worked out at the appropriate points of the range of integration. A system of linear algebraic equations is generated by giving different values to X_0 and solved for the Fourier coefficients by the least-square-error method. Numerical values of the wake vortex intensity for different ω (scaled on the constant speed of propagation of the tail) for rectilinear motion obtained from the Fourier coefficients for different positions of X_1, through (27), agree with the values obtained from equation (76) of Lighthill's (1970) analysis, which adds to the confidence in the method. The merit of this method over the earlier theories of Lighthill (1970) and Wu (1971) is that it is capable of giving a quantitative estimate of the vortex intensity of the wake shed by an aerofoil making finite amplitude oscillations. Now that the wake vorticity is known at any instant, the vortex intensity induced by the complete wake, using (11), is given by

$$\gamma_1(X) = \frac{1}{2\pi}\int_{X_0}^{\infty} \frac{\gamma(X_1)\sec\theta_1}{[c^2-(c+x)^2]^{\frac{1}{2}}} f(X_1, X_0)\left\{2-\beta\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\right\} dX_1$$

and the vortex intensity induced on the aerofoil by the angle of incidence is given by (5), where x is related to X, on the aerofoil, through

$$x = (X - X_0)\sec(\theta-\alpha).$$

The component of the impulse in the Y direction is

$$Q = \rho\int_{X_0-2c\cos(\theta-\alpha)}^{X_0} X\{\gamma_0(X)+\gamma_1(X)\}\sec(\theta-\alpha)\,dX + \rho\int_{X_0}^{\infty} X_1\gamma(X_1)\sec\theta_1\,dX_1. \tag{28}$$

By making use of the integrations

$$\int_{-2c}^{0} \frac{[2-\beta(\tau_1^{-1}+\tau_2^{-1})]}{[c^2-(c+x)^2]^{\frac{1}{2}}}\,dx = 2\pi\left(1-\frac{\beta}{\alpha_1}\right)$$

and

$$\int_{-2c}^{0} \frac{(c+x)[2-\beta(\tau_1^{-1}+\tau_2^{-1})]}{[c^2-(c+x)^2]^{\frac{1}{2}}}\,dx = -\frac{2\pi\beta}{\alpha_1}\frac{2a^2x_1'}{x_1'^2+y_1'^2}$$

it can be worked out that

$$\int_{X_0-2c\cos(\theta-\alpha)}^{X_0} X\gamma_0(X)\sec(\theta-\alpha)\,dX = \Gamma_0\{X_0-\tfrac{3}{2}c\cos(\theta-\alpha)\} + \tfrac{1}{2}\pi c^2\dot{X}_0 D_0\cos(\theta-\alpha)$$

and

$$\int_{X_0-2c\cos(\theta-\alpha)}^{X} X\gamma_1(X)\sec(\theta-\alpha)\,dX = \int_{X_0}^{\infty}\gamma(X_1)\left\{X_c\delta_1 - \frac{acx_1'}{x_1'^2+y_1'^2}\cos(\theta-\alpha)\right\}\sec\theta_1\,dX_1,$$

where

$$D_0 = \omega\alpha_2\cos(\omega X_0/c), \quad \delta_1 = \alpha_1/\beta - 1$$

and

$$\Gamma_0 = \pi c U(D_0 - 2\alpha\sec\theta).$$

Substituting back for these expressions in (28) we get

$$Q = \rho\left[\Gamma_0\{X_0-\tfrac{3}{2}c\cos(\theta-\alpha)\} - \frac{\pi c^2}{2}D_0 U\cos(\theta-\alpha) + \int_{X_0}^{\infty} X_1\gamma(X_1)\sec\theta_1\,dX_1 + \int_{X_0}^{\infty}\gamma(X_1)\left\{X_c\delta_1 - \frac{cax_1'}{x_1'^2+y_1'^2}\cos(\theta-\alpha)\right\}\sec\theta_1\,dX_1\right]$$

and (15) gives, on making use of the result

$$d\Gamma_0/dX_0 + d\Gamma_1/dX_0 = \gamma(X_0)\sec\theta,$$

$$L_Y = \rho c U^2\Big[-\tfrac{1}{2}\pi\{D_2\cos(\theta-\alpha) - D_0^2\sin(\theta-\alpha)\} + D_1\{1+\tfrac{3}{2}D_0\sin(\theta-\alpha)\} - \tfrac{3}{2}D_3\cos(\theta-\alpha) + \int_{X_0}^{\infty}\frac{\gamma(X_1)}{U}\delta_1\sec\theta_1\,dX_1 - \frac{d}{dX_0}\int_{X_0}^{\infty}\frac{\gamma(X_1)}{U}(\delta_1+\delta_2)\cos(\theta-\alpha)\sec\theta_1\,dX_1\Big], \quad (29)$$

where $D_1 = \pi[2\alpha\sec\theta - \omega\alpha_2\cos(\omega X_0/c)]$,

$$D_2 = -\omega^2[\alpha_2\sin(\omega X_0/c) + 2\omega^2h^2\sin\theta\cos^3\theta\cos^2(\omega X_0/c)],$$

$$D_3 = \pi[D_2 + 2\alpha_0\omega\{\sec\theta - h\omega\sin\theta\sin(\omega X_0/c)\}\cos(\omega X_0/c)]$$

and $\delta_2 = ax_1'/(x_1'^2 + y_1'^2)$.

The suction force experienced by the caudal fin per unit span is given by (18) as

$$F_s = -\pi\rho c U^2\Big[2^{\frac{1}{2}}(D_0 + \alpha\sec\theta) - \frac{1}{2\pi}\int_{X_0}^{\infty}\frac{\gamma(X_1)\sec\theta_1}{U}f_1(X_1, X_0)\,dX_1\Big]^2, \quad (30)$$

where the integral in (30) expresses the reduction in the forward thrust because the wake vortices induce a downwash on the aerofoil which reduces the intensity of the fast flow around the leading edge. The suction force obtained for the steady case and the small amplitude case agree with the results existing in the literature. The thrust coefficient C_T, defined as the thrust per unit area divided by $\frac{1}{2}\rho U^2$, using (29) and (30) in (19), is given by

$$C_T = \bar{L}_X/\rho c U^2. \quad (31)$$

It may be pointed out that the range of integration of the integrals occurring in (29) and (30) may be reduced to a period only, by using the periodicity of $\gamma(X_1)\sec\theta_1/U$, and that the convergence of δ_1, $\delta_1+\delta_2$ and $f_1(X_1, X_0)$ may be ensured by subtracting suitable constants on the same lines as in the case of $f(X_1, X_0)$. These values are

$\omega\sec(\theta-\alpha)/2\pi n$ in the case of δ_1,

$3\omega\sec(\theta-\alpha)/2\pi n$ in the case of $\delta_1+\delta_2$

and $\omega\sec(\theta-\alpha)/2^{\frac{1}{2}}\pi n$ in the case of $f_1(X_1, X_0)$.

The singularity at the lower limit is again removed by the substitution

$$X_1 = X_0 + u^2$$

and the differentiation with respect to X_0 in (29) is carried out numerically.

The hydromechanical propulsive efficiency is calculated from energy considerations as outlined in an earlier section. For this particular motion, the kinetic energy of the wake, which consists of an integral number of wavelengths, say n, is given by

$$T = -\frac{1}{4\pi}\rho U^2 \int_0^{nX_P}\int_0^{nX_P} \frac{\gamma(X_1)}{U}\frac{\gamma(X_2)}{U}\log r \sec\theta_2 \sec\theta_1\, dX_2\, dX_1.$$

If the tail moves through another wavelength, the kinetic energy imparted to the fluid is given by

$$E = -\frac{1}{4\pi}\rho U^2 \int_0^{X_P} \frac{\gamma(X_1)\sec\theta_1}{U}\left\{\int_{-\infty}^{\infty} \frac{\gamma(X_2)}{U}\sec\theta_2 \log r\, dX_2\right\} dX_1.$$

The value of E can be obtained using generalized functions as shown below.

The integral in the curly brackets, I say, can be expressed as

$$I = \int_{-\infty}^{\infty} \frac{\gamma(X_2)\sec\theta_2}{U}\log(X_1 - X_2)\, dX_2 + \int_{-\infty}^{\infty} \frac{\gamma(X_2)\sec\theta_2}{U}\log\left\{1 + \left(\frac{Y_1 - Y_2}{X_1 - X_2}\right)^2\right\}^{\frac{1}{2}} dX_2.$$

The integrand of the second part is a perfectly well behaved function whereas the first part, I_2 say, can be written as

$$I_2 = \sum_{m=1}^{\infty}\left[a_m \int_{-\infty}^{\infty} \cos\left(\frac{2m\pi X_2}{X_P}\right)\log(X_1 - X_2)\, dX_2 + b_m \int_{-\infty}^{\infty} \sin\left(\frac{2m\pi X_2}{X_P}\right)\log(X_1 - X_2)\, dX_2\right]$$

$$= -\sum_{m=1}^{\infty}\left\{\frac{a_m X_P}{2m}\cos\left(\frac{2m\pi X_1}{X_P}\right) + \frac{b_m X_P}{2m}\sin\left(\frac{2m\pi X_1}{X_P}\right)\right\}$$

because $$\int_{-\infty}^{\infty} \cos\frac{2m\pi X_1}{X_P}\log|X - X_1|\, dX_1 = -\frac{X_P}{2m}\cos\left(\frac{2m\pi X}{X_P}\right)$$

and $$\int_{-\infty}^{\infty} \sin\frac{2m\pi X_1}{X_P}\log|X - X_1|\, dX_1 = -\frac{X_P}{2m}\sin\left(\frac{2m\pi X}{X_P}\right).$$

[(See Lighthill 1959, p. 39.) The Fourier transform

$$\int_{-\infty}^{\infty} \exp(-2\pi i x y)\log|x|\, dx = -\tfrac{1}{2}|y|^{-1},$$

which with $y = m/X_P$ and $x = X_2 - X_1$ gives

$$\int_{-\infty}^{\infty} \exp\left(-2\pi i X_2 \frac{m}{X_P}\right)\log|X_1 - X_2|\, dX_2 = -\frac{1}{2}\left|\frac{X_P}{m}\right|\exp\left(-2\pi i X_1 \frac{X_P}{m}\right).\Big]$$

Thus

$$W_E = \frac{1}{4\pi}\rho U^2 \int_0^{X_P}\left[\sum_{n=1}^{\infty} a_n \cos\frac{2n\pi X_1}{X_P} + b_n \sin\frac{2n\pi X_1}{X_P}\right] \times\left[\sum_{m=1}^{\infty}\frac{X_P}{2m}\left(a_m \cos\frac{2m\pi X_1}{X_P} + b_m \sin\frac{2m\pi X_1}{X_P}\right)\right] dX_1$$

$$-\frac{1}{4\pi}\rho U^2\int_0^{X_P}\frac{\gamma(X_1)\sec\theta_1}{U}\int_{-\infty}^{\infty}\frac{\gamma(X_2)\sec\theta_2}{U}\log\left\{1+\left(\frac{Y_1-Y_2}{X_1-X_2}\right)^2\right\}^{\frac{1}{2}}dX_2\,dX_1$$

$$=\frac{1}{4\pi}\rho c^2U^2\left[\frac{X_P^2}{4c^2}\sum_{n=1}^{\infty}\frac{a_m^2+b_m^2}{n}\right.$$

$$\left.-\int_0^{X_P}\frac{\gamma(X_1)\sec\theta_1}{U}\int_{-\infty}^{\infty}\frac{\gamma(X_2)\sec\theta_2}{U}\log\left\{1+\left(\frac{Y_1-Y_2}{X_1-X_2}\right)^2\right\}^{\frac{1}{2}}\frac{dX_2}{c}\frac{dX_1}{c}\right].$$

The contribution to W_E from the double integral can be obtained numerically and the hydromechanical efficiency can be obtained from

$$\text{efficiency} = -U\bar{L}_X/(-U\bar{L}_X+\overline{W}_E),$$

where

$$\overline{W}_E = \frac{1}{X_P}\int_0^{X_P} W_E\,dX_1.$$

The hydromechanical efficiency of a lunate tail traversing a straight path with uniform velocity and executing pitching oscillations about the trailing edge is given by Lighthill's (1970) equation (78) to be

$$1-\frac{(\omega^2+4)(F-F^2-G^2)}{\omega^2(3F-1)-6\omega G}$$

(where F and G are the real and imaginary parts of the Theodorsen (1935) function), whose values for different reduced frequencies agree with the results obtained by our procedure. It may be pointed out that the value of the efficiency is negative in this case. We know that the thrust is positive (in the $-X$ direction) if the energy imparted to the wake per unit time is less than the rate of working in maintaining the oscillations, but in this case the mean rate of shedding of energy into the wake is greater than the mean rate of working because part of the energy is being supplied from the fluid. The thrust generated is along the $+X$ axis and denotes resistance or drag. This type of motion is never observed in nature and is studied here only as a check on our analysis.

Another verification of the correctness of this method is carried out through comparison with small amplitude motion. The vertical displacement of the section of the caudal fin stretching along the x axis from $-2c$ to 0 is given by

$$Y = Y_0+(X-X_0)\tan(\theta-\alpha).$$

Assuming that the amplitude of the path is small

$$\tan(\theta-\alpha) \sim \theta-\alpha = -h\omega\sin(\omega X_0/c)+\alpha_0\sin(\omega X_0/c).$$

Thus

$$Y = h_1\cos(\omega X_0/c)+(\alpha_0-\omega h)(X-X_0)\sin(\omega X_0/c)$$

and the feathering parameter, defined as the ratio of the slope of the plate to the slope of the path of the pitching axis (in our case the trailing edge), is given by

$$\frac{\tan(\theta-\alpha)}{\tan\theta} \sim \frac{\theta-\alpha}{\theta} = 1-\frac{\alpha_0}{\omega h}.$$

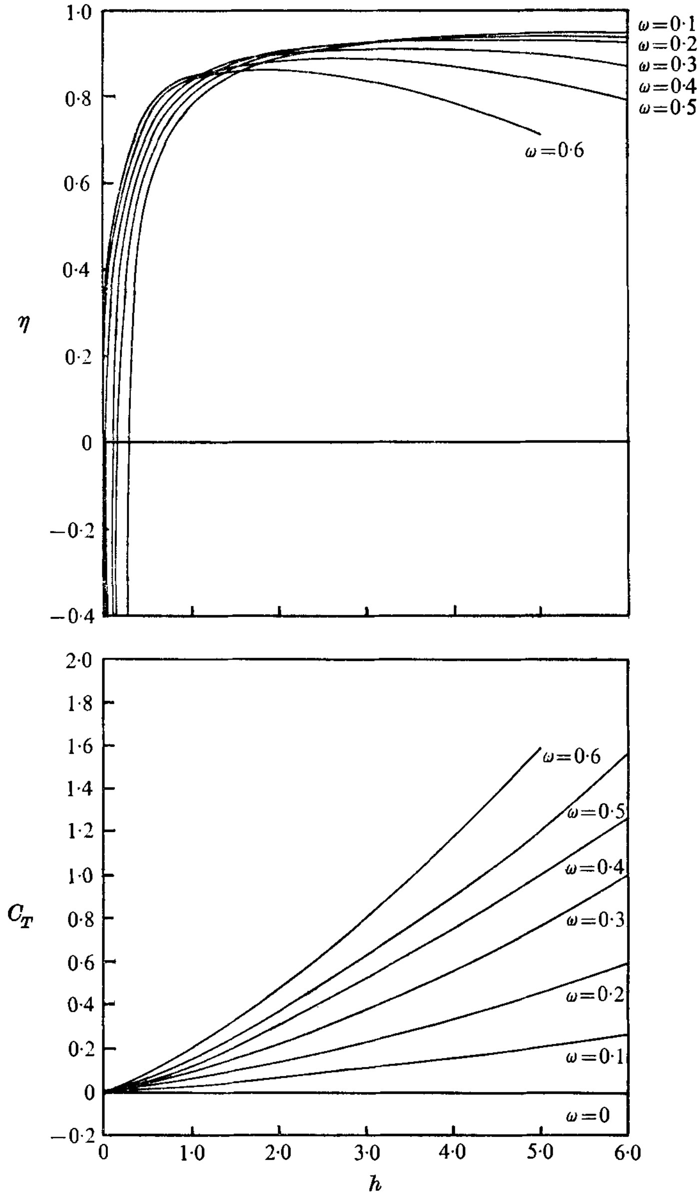

FIGURE 2. Thrust coefficient C_T and efficiency η predicted by two-dimensional finite amplitude aerofoil theory for values of 0·1, 0·2, 0·3, 0·4, 0·5 and 0·6 of the reduced frequency plotted as functions of h ($= h_1/c$, i.e. the path amplitude scaled on tail's semi-chord length) for (*a*) $\alpha_0 = 0{\cdot}16$ rad, (*b*) $\alpha_0 = 0{\cdot}24$ rad, (*c*) $\alpha_0 = 0{\cdot}32$ rad.

The feathering parameter for $\omega = 0{\cdot}2$, $h = 1$ and $\alpha_0 = 0{\cdot}16$ rad turns out to 0·2 and the hydromechanical propulsive efficiency for the motion specified by these parameters is 0·80, which agrees with the value obtained by Lighthill (1970). These comparisons add to the confidence in these finite amplitude lunate-tail hydrodynamics.

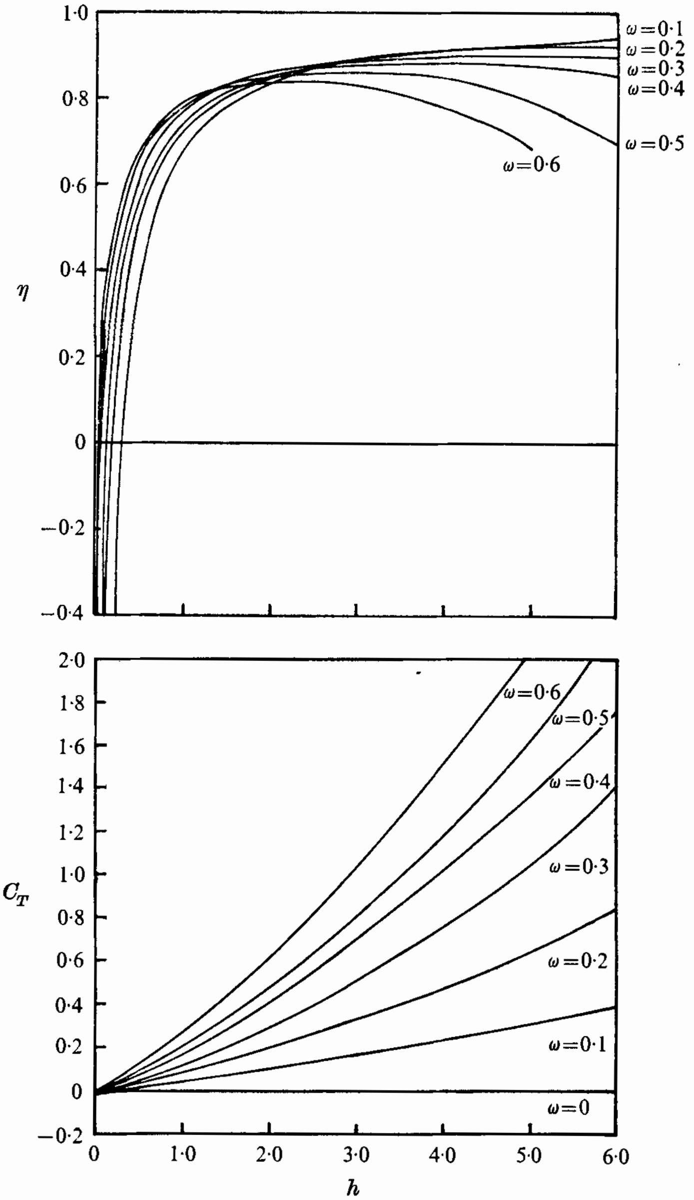

FIGURE 2(*b*). For legend see p. 178.

Numerical computations have been carried out in order to have a quantitative estimate of the dependence of the thrust and efficiency on the amplitude of the motion, the angle of attack and the reduced frequency. The thrust coefficient and the efficiency are plotted in figure 2 against the amplitude scaled on the semi-chord length of the lunate tail for different values of the reduced frequency. This clearly brings out the amplitude effect, which is one of the main motivations of this study. The effect of the variation of the angle of attack has also been

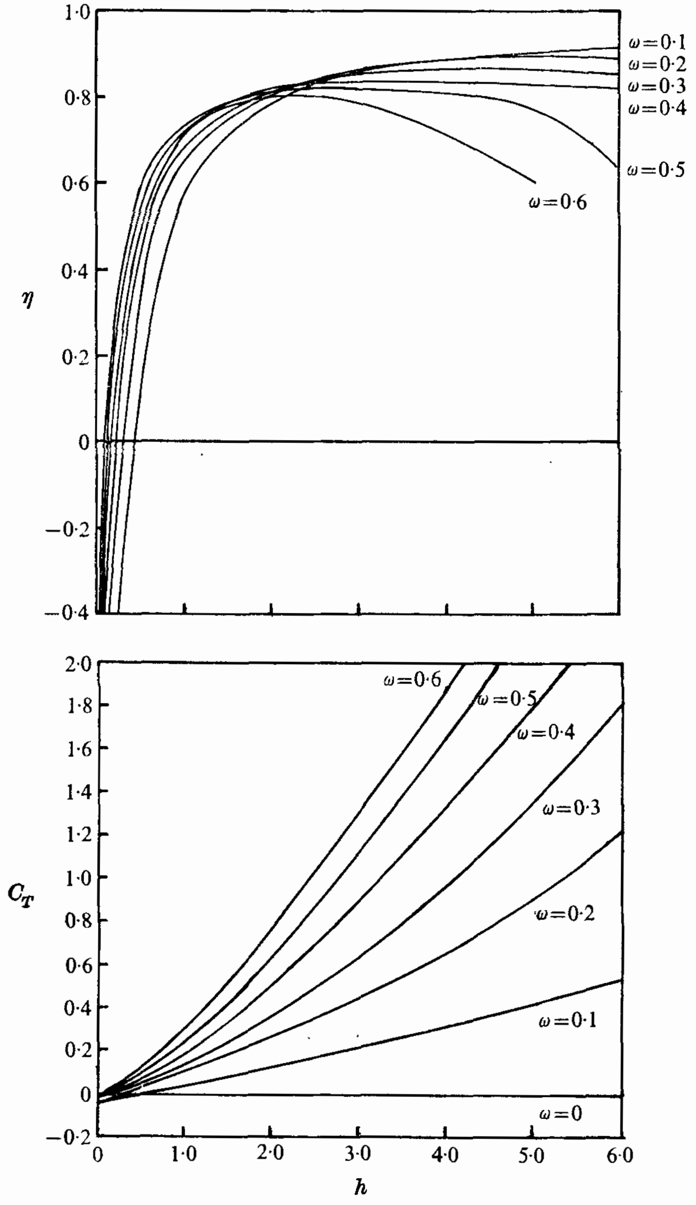

FIGURE 2(*c*). For legend see p. 178.

depicted in figures 2(*a*)–(*c*), which are for $\alpha_0 = 0{\cdot}16$, $0{\cdot}24$ and $0{\cdot}32$ rad respectively. The tail is assumed to be pitching about the trailing edge primarily because Lighthill's (1970) analysis shows preference for the pitching axis to be close to the trailing edge for optimum values of the thrust and efficiency. However, our analysis, although a little more involved computationally, can be developed for other positions of the pitching axis without excessive complications.

For lower values of the reduced frequency, viz. $\omega = 0{\cdot}1$, the efficiency increases with increases in the amplitude with a peak near $h = 10$–12 depending on the angle of attack, but as the unsteady effects become more and more important the peak values of the efficiency, although reduced in magnitude, are realized at lower values of the amplitude. It is interesting to note that for intermediate values of the reduced frequency (0·3–0·5) there is no sharp maximum and the lunate-tailed animal can operate over a large range of amplitudes with fairly high values of the efficiency, dictated by the thrust requirements. Still higher values of the reduced frequency result in a comparatively steeper rise and fall in the efficiency, leaving the animal with a very limited range of efficient operational amplitudes.

Lunate-tailed animals are, essentially, high-speed animals and, therefore, must achieve high thrust values besides realizing them efficiently. The thrust coefficient C_T based on tail area and swimming speed must be large enough to balance the drag coefficient based on the animal's *total* surface area and on the swimming speed. We see that this requires a suitable combination of a sufficiently high amplitude h and sufficiently high frequency parameter ω. The thrust coefficient shows marked increases also with increases in the angle of attack but the energy imparted to the fluid increases too, resulting in a decrease in the propulsive efficiency. This increase in the thrust as α_0 increases is entirely due to a steep increase in the leading-edge suction force, but realization of high thrust values with largish values of α_0 is not advisable because of the likelihood of leading-edge separation, which results in a substantial reduction of the thrust as well as the efficiency. These considerations suggest that lunate-tailed animals may be forced to undulate their tails at largish amplitudes and frequencies with a lower angle of attack to ensure the requisite thrust values accompanying high speeds. These remarks are borne out by figures 2 (*a*)–(*c*), which compare the variation of thrust and efficiency with the amplitude, the reduced frequency and the angle of attack.

The most efficient swimming is with large amplitudes accompanied by very small values of the angle of attack and the reduced frequency but the thrust values then realized are very low, whereas for higher values of the angle of attack and reduced frequency accompanied by high values of the amplitude the thrust values are enhanced but the efficiency is lowered. Thus the swimming animal has to strike a balance between the competing trends of thrust and efficiency and undulate its tail appropriately to realize those values of the angle of attack, the amplitude and the reduced frequency which give optimum swimming performance. This investigation suggests a preference for reduced frequencies around 0·4 accompanied by an amplitude as high as twice the mean chord length of the lunate tail, which seems to be in line with the data available from the carefully documented experiments of Fierstine & Walters (1968) on wavyback skipjacks.

This theory overestimates the values of the thrust and the efficiency as the streamwise wake vorticity, resulting from the finiteness of the lunate tail, has not been taken into account. The analysis for a finite lunate tail performing finite amplitude oscillations can be developed, for a model rectangular shape, using the ideas of Sears (1938) and Chopra (1974). A crescent-moon-shaped tail performing small amplitude oscillations is being studied by Chopra & Kambe, using lifting-surface theory, with the aim of discussing the shape of the vortices shed into the wake and their relevance to Lighthill's (1970) principal suggestion, that a crescent-moon shape of the tail seems to be the culminating point of the evolution of the fast-moving aquatic animals in the enhancement of speed and hydromechanical efficiency because it may assist shedding of circular vortex rings, which in general are very good devices for carrying as much momentum as possible in relation to their energy. However, the problem of crescent-moon-shaped tails performing finite amplitude oscillations still needs careful consideration.

The author acknowledges with gratitude the benefit of suggestions and comments from Professor Sir James Lighthill, F.R.S., Lucasian Professor. Thanks are also due to Professor T. Y. Wu for his interest and helpful discussions on some aspects of the problem and to the Association of the Commonwealth Universities for a post-doctoral award.

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